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Optimal maintenance of deteriorating equipment using semi-Markov decision processes and linear programming

G.A. Kechagias[a], A.C. Diamantidis[a,*], T.D. Dimitrakos[b], M. Tsakalerou[c]

[a] *Aristotle University of Thessaloniki, Faculty of Economics and Political Sciences, Thessaloniki, Greece;*
[b] *University of the Aegean, Department of Mathematics, Samos, Greece;*
[c] *Nazarbayev University, Department of Civil and Environmental Engineering, Astana, Kazakhstan*

ABSTRACT

This paper considers a mathematical model analysing the deterioration of system equipment and available maintenance options. Under specific conditions on costs and transition probabilities of the model, the issue of ideal maintenance of the equipment by assuming that preventive maintenance, condition-based maintenance, and the replacement times of the equipment follow known continuous probability distributions is explored. A semi-Markov decision process formulation is provided for this model and computational analysis is possible by applying well-known Markov decision algorithms. A linear programming approach is also presented with appropriate constraints established for the equipment's maintenance and repair times. Various numerical results are also presented for the validation of the model. The motivation for this paper is to develop a realistic framework for determining decisions about the type of maintenance, possible replacement, or the continuation of operation of deteriorating equipment. The above decisions are taken within a framework of time constraints for equipment maintenance and replacement. Such limitations in the implementation of these actions are very crucial for all maintenance management systems. The purpose of this paper is to make the described model a useful tool for maintenance managers who plan and implement optimal maintenance policies within an environment of variable time constraints.

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*Corresponding author:
A.C. Diamantidis
adiama@econ.auth.gr

1. Introduction

Many articles have appeared dealing with the study of systems in which the management of their equipment is a major concern. Equipment is a technological object, which is a part of the process of a system. This could be, for example, a motor grader in the construction process of a project by a construction company or a machine which assembles vehicle parts in an automotive industry. The complete repair or maintenance of system equipment is crucial to the system's performance quality. During the previous decades, maintenance was considered an appropriate action that may be chosen after an equipment failure. Nowadays, maintenance is established as a crucial element in asset management. Organizations can enhance their functionality and reliability by planning

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maintenance. So, the maintenance of systems equipment influences the overall system efficiency. Mainly, in literature, various general types of maintenance philosophies can be identified, namely, among others, preventive, corrective, condition-based, and risk-based. The first type is to preventively maintain the equipment to a better working condition than the one the equipment has at a specific time epoch. This kind of maintenance is called Preventive Maintenance (PM) of the equipment. The second type is to completely replace (or completely repair) the equipment when this system component fails to operate at all. This kind of maintenance is called Replacement (R) or Corrective Maintenance (CM) of the equipment. Condition-Based Maintenance (CBM) is considered as a more advanced alternative to PM. Risk-Based Maintenance (RBM) philosophy provides a tool for maintenance scheduling and decision-making aimed at reducing the possibility of equipment failure.

The Markov decision process is an appropriate mathematical tool that is frequently used to analyze the deterioration of an operating device. The general approach is as follows: A system (e.g., simple machine or production line) that deteriorates over time, is monitored by a possible inspector who may choose an action from a set of alternative actions (e.g., preventive maintenance, repair, continuation of operation). One's objective is to find an optimal decision policy to minimize a predefined function of the total expected cost of the system. Possible techniques for determining the optimal policy are usually related to the method of dynamic programming and its standard or modified algorithms.

In many such models, it can be shown that the optimal policy initiates a kind of equipment maintenance if the level of its deterioration exceeds a critical level. Such a policy is called a control-limit policy, and the critical level is called the control-limit. Some articles on research and results in this area with the application of Markov decision models are reported.

Tijms and Duyn Schouten [1] presented a Markov decision algorithm operating on the category of control-limit policies for deteriorating equipment. Under a cost structure, they determined an optimal schedule for equipment management. Makis and Jardine [2] considered a replacement model with general repair which brings the equipment to a better state. They formulated a semi-Markov decision model in which the equipment failure is the decision epoch. A two-dimensional infinite state space was considered, where the state consists of the number of failures and the age of the system. Love et al. [3] reformulated the previous model with a finite-state semi-Markov decision model, by truncating the state space and by discretizing the second state variable of the system age. They developed an algorithm that creates a sequence of control-limit policies to determine the optimal replacement policy. Dimitrakos and Kyriakidis [4] improved this algorithm by applying Tijms's [5] embedding technique. This technique reduces the calculations of the algorithm which have been augmented significantly due to the discretization of the system age variable. They developed an efficient policy iteration algorithm that generates a sequence of control-limit policies intending to find the optimal replacement strategy. Vanneste and Duyn Schouten [6] formulated a finite-state Markov decision model for ideal maintenance in a flow line with a buffer in the middle. They proved that, for each buffer level, the optimal policy is of a control-limit type. Kyriakidis and Dimitrakos [7] provided a generalized infinite-state Markov decision formulation for this model. They proved the existence of the optimal policy and that it is of a control-limit type. Chen and Trivedi [8] presented an approach to optimizing the fixed inspection rate maintenance policy using a semi-Markov decision process. The model exports the best combinations of inspection rate and maintenance policy. Lugtigheid et al. [9] presented an equipment replacement decision model. They developed a continuous-time Markov decision process in which the aim was to define a policy that minimizes the total cost. The system component can be replaced at any point or may be repaired at failure or may be preventively maintained. Tomasevicz and Asgarpoor [10] formulated a Markov decision model to find the maintenance strategy for repairable power equipment. Their approach determines if maintenance should be implemented for the equipment and what kind of maintenance, among preventive maintenance, minor maintenance, and major maintenance should be adopted in each deterioration state. Pavitsos and Kyriakidis [11] considered a system with an input generator, an intermediate buffer, and a production unit. The production unit deteriorates over time and the issue of its optimal maintenance strategy was studied. They proved that for a fixed buffer size if the repair times are geometrically distributed, then the optimal policy is control-limit type. Manatos et al. [12] formulated a repairable production system in which PM is performed to improve the system's reliability and performance. They determined the optimal inspection times and the optimal maintenance strategies to optimize the system's measures. Liu et al. [13] developed a maintenance policy for systems involving aging and cumulative damage. The second feature was modeled by a degradation process, and two cases were considered. The two cases are determined by whether the distribution parameters of this process are known or not. Liu et al. [14] proposed a maintenance strategy for a degrading system with aging and state-dependent operating costs. They developed a model to investigate the optimal replacement policy and they extended this to a model in which imperfect repair restores the system to its operating condition. Liu et al. [15] presented a maintenance strategy for a multi-component system involving hidden failures. These failures can be observed at system inspection. The aim was to establish the optimal inspection intervals for each component to minimize the long-run cost rate. Si et al. [16] developed a covariate-dependent trend renewal process model to formulate the heterogeneous failure process of multiple systems by utilizing a semi-Markov decision process and proposed a two-state covariate-dependent optimal maintenance schedule for these systems. Guo and Gu [17] analyzed the joint decision-making of production and maintenance strategies in mixed-model assembly systems. They formulated this model as a Markov decision process that minimizes the average cost and Monte Carlo simulation was applied to estimate the system's performance measures under the optimal policy. Wang and Miao [18] studied a reliability problem for the optimal maintenance of balanced systems. They formulated a semi-Markov preventive maintenance optimization model for a balanced system with n identical units, where each unit involved degradation failure and sojourn times following the Erlang distribution with different parameters for different function zones. Yang et al. [19] formulated a finite-horizon Markov decision process for the maintenance of single-machine scheduling problems, where the processing times of jobs are based on their position in the production sequence. Their aim was the optimization of the machine make-span. The effectiveness of their approach was demonstrated by computational experiments.

Another approach to maintenance scheduling is through the application of heuristic and meta-heuristic algorithms. Such approaches are presented in the papers of Rastgar et al. [20], Nahas et al. [21], and Sharifi and Taghipour [22]. Other maintenance management techniques such as Analytic Hierarchy Process and Causal Tree Analysis are presented in the papers of Lopes et al. [23], and Murad et al. [24].

2. Paper contribution and research objectives

In most research efforts approached with Markov decision processes, the actions in each deterioration state usually involve only one type of maintenance. The other actions are usually replacements or repairs that bring the equipment back to working condition as Good as New. However, maintenance managers usually must choose between two, different cost and different impact, maintenances that do not necessarily bring the system to the state as Good as New. The proposed model in this paper aims to fill this research gap by incorporating two maintenances actions with realistic and applicable manner, that both can bring the system to every state of lower degree deterioration, have different costs, and different transitions probabilities to these states. Another issue that has not been sufficiently studied in the literature is the study of systems with the addition of constraints on maintenance and repair times. This condition is very common in maintenance systems. Many times, maintenance is required by various factors to be completed within specific time limits. The contribution of this paper in this research gap is the solution of the model using linear programming. This method allows the introduction of constraints on maintenance and replacement times, and modifies the optimal policy and cost based on these constraints.

The proposed model (Table 1) can be considered as a combination of the models Chen-Trivedi [8] and Kyriakidis-Dimitrakos [7]. The following table lists the main characteristics of the semi-Markovian approximation of Chen-Trivedi. In this model, we incorporate the cost structure conditions of Kyriakidis-Dimitrakos [7] model, and the characteristics related to the research gap that the present work seeks to approach.

Table 1. A brief summary of Chen-Trivedi model

Year:	*Authors:*	*States of device:*	*Actions:*	*Solving Methods:*
2005	Chen, Trivedi	$\{0,1,\ldots,k+1\}$	• 0: no action is taken • 1: minimal maintenance • 2: major maintenance (as good as new)	Value iteration algorithm

The present paper is mainly concerned with a semi-Markov decision process (SMDP) formulation to analyze the deterioration process and available maintenance strategy of system equipment. The cost structure of the model incorporates operating, preventive maintenance, replacement, and condition-based maintenance costs and it is assumed that all these costs are based on the working condition of the equipment. The contribution of this paper regarding the investigation of the optimal maintenance strategy problem of system equipment is the following. Under a specific cost structure and reasonable assumptions on the transition probabilities of equipment, its stochastic deterioration process is modeled using a SMDP. Various numerical examples provide strong evidence that the optimal policy is of a threshold type, and it is characterized by two critical numbers. A linear programming formulation approach is also provided for the model to solve the problem under appropriate constraints that can be established for the maintenance and repair times of the equipment. To our knowledge, there is no other application in the literature to solve the expected discounted cost problem for a semi-Markov decision process using linear programming. A comparison between our basic model and a simplified model that has the possibility only for preventive maintenance and equipment replacement only in a specific degree of its deterioration, which is the most common practice in scheduling system equipment maintenance, is presented.

The rest of the paper is organized in the following way. The description of the model and its formulation is provided in Section 3. In Section 4, the dynamic programming algorithms for determining the optimal policy are given. In Section 5, a linear programming formulation is also presented with appropriate constraints established for the equipment's maintenance and replacement times. In Section 6, numerical results are presented for the described model. In Section 7, a comparison between the model and a simplified one which is used more in reality is given. Finally, Section 8, provides conclusions and direction for future research based on the limitations of the present model.

3. Description of problem and conditions

For the description of the model in this section, a summary of its parameter notations is given in Table 2.

A system's equipment that deteriorates over time is considered. It may be assumed that the equipment could be a component that is critical for the functionality of a processing machine that belongs to a production line. It is supposed that i, $i \in \{0,\ldots,m,m+1\}$ represents the working condition of the equipment, where the equipment is found to be into one of $m+2$ working conditions $0,1,\ldots,m+1$, which describes the increasing degree of equipment deterioration. State 0 denotes new equipment or an old one that functions as well as new, whereas state $m+1$ denotes failed or non-operative equipment. The states $1,2,\ldots,m$ are operative. It is assumed that, leaving the equipment to operate, it deteriorates as time evolves. The state space of the system is expressed as:

$$S=\{0,\ldots,m,m+1\}$$

Equipment is observed at discrete equidistant time epochs and an action must be made at each epoch. The equipment can also be maintained, and a decision must be made at the end of each maintenance. The system can replace the equipment if its deterioration degree is equal to or exceeds a specific

Table 2. Notation of model parameters

S	*State space.*
w	*State from which the equipment can be replaced.*
$p_{ij}(a)$	*Probability transition from state i to state j when action a is chosen.*
c_i	*Operating cost until the next time epoch.*
C_{PM}	*Intermediate preventive maintenance cost.*
$c_{PM}(i)$	*Preventive maintenance cost rate.*
C_{CBM}	*Intermediate condition-based maintenance cost.*
$c_{CBM}(i)$	*Condition-based maintenance cost rate.*
C_R	*Intermediate replacement cost.*
$c_R(i)$	*Replacement cost rate.*
m_{PM}	*Expected time for preventive maintenance.*
m_{CBM}	*Expected time for condition-based maintenance.*
m_R	*Expected time for replacement of the equipment.*

value w. There are four alternative actions. Action 0 (leave the equipment to operate), Action $1A$ (start preventive maintenance of the equipment), Action $1B$ (start condition-based maintenance of the equipment), and Action 2 (replace the equipment). At state 0 the only possible action is action 0 and at state $m+1$ the action 2. At states $i, w \leq i \leq m$ actions $0,2,1A$ and $1B$ are possible. At states $i, 1 \leq i \leq w$-1, actions $0,1A$ and $1B$ are possible.

If the equipment is at state i, $1 \leq i \leq m$, and action 0 of leaving the equipment to operate is chosen, then at the next time epoch, the equipment may be found to be at any state $j \geq i$, with probability $p_{ij}(0)$. It is assumed that for each i, $p_{ij}(0) > p_{i,j+1}(0)$, $i \leq j \leq m+1$. If the equipment is at state i, $1 \leq i \leq m$, and action $1A$ of PM is chosen, then at the next time epoch, the equipment may be found to be at any state j with probability $p_{ij}(1A)$. It is also assumed that for each i, $p_{ij}(1A) > p_{i,j-1}(1A)$, $0 \leq j \leq i$-1. If at state i, $1 \leq i \leq m$ action $1B$ of condition-based maintenance is chosen, then at the next time epoch, the equipment may be found to be at any state j with probability $p_{ij}(1B)$. It is supposed that for each i, $p_{ij}(1B) < p_{i,j-1}(1B)$, $0 \leq j \leq i-1$. Preventive and condition-based maintenance of the equipment cannot be disrupted and redirect the equipment to any «better» condition j, where $j \in \{0,1,..., i-1\}$. It is further assumed that, if the equipment is considered to be at condition i and action $1A$ is chosen, then given that the next working condition is j, the preventive maintenance time, until the transition from i to j occurs, is a continuous random variable with known probability distribution $F_{ij}^{PM}(t)$. Respectively, if the equipment is at working condition i and action $1B$ is chosen, then given that the next working condition is j, the condition-based maintenance time, until the transition from i to j occurs, is a continuous random variable with known probability distribution $F_{ij}^{CBM}(t)$. It is further assumed that, if the equipment is considered to be at state $i, w \leq i \leq m+1$, and action 2 is chosen, then given that the next state is 0, the replacement time until the transition from i to 0 occurs is a continuous random variable with known probability distribution $F_{i0}^{R}(t)$. If the action of replacement is chosen, then the equipment is replaced by an entirely new one.

If the equipment considered to be at state i, $0 \leq i \leq m$ and action 0 of leaving the equipment to operate is chosen, then an operating cost equal to c_i is incurred until the next time epoch. If the equipment considered to be at state i, $1 \leq i \leq m$ and action $1A$ of preventive maintenance is chosen, then an intermediate preventive maintenance cost is incurred which is equal to C_{PM} and a preventive maintenance cost rate $c_{PM}(i)$ is imposed until the next transition occurs. If the equipment is at state i, $1 \leq i \leq m$ and action $1B$ of condition-based maintenance of the equipment is chosen, then an intermediate condition-based maintenance cost is incurred which is equal to C_{CBM} and a condition-based maintenance cost rate $c_{CBM}(i)$ is imposed until the next transition occurs. If the equipment considered to be at states i, $w \leq i \leq m+1$ and action 2 of replacement is chosen, an intermediate replacement cost is incurred which is equal to C_R and a replacement cost rate equal to $c_R(i)$ is imposed until the next transition occurs. Let also m_{PM}, m_{CBM}, and m_R be the expected time for preventive maintenance, condition-based maintenance, and a replacement of the equipment, respectively. The following reasonable conditions on the cost structure and the transition probabilities of the model may be imposed.

Condition 1: For working condition $i = 1,...,m$, it is assumed that: $c_{PM}(i) \leq c_{PM}(i+1)$ and $c_{CBM}(i) \leq c_{CBM}(i+1)$. That is, as the equipment deteriorates, the preventive maintenance and the condition-based maintenance cost rate increase.

Condition 2: For working condition i of the equipment, the preventive maintenance cost rate is smaller than its condition-based maintenance and its replacement cost rate, i.e., it is assumed that: $c_{PM}(i) \leq c_{CBM}(i) \leq c_R(i)$.

Condition 3: Regarding the expected times it is assumed that: $m_{PM} < m_{CBM} < m_R$.

Condition 4: For each condition i of the equipment, its intermediate replacement cost is greater than its intermediate condition-based maintenance cost and its intermediate preventive maintenance cost, i.e., it is assumed that: $C_{PM} < C_{CBM} < C_R$.

Condition 5: For each condition $i=0,...,m$, it is assumed that: $c_i \leq c_{i+1}$ i.e., as equipment deteriorates, the operating cost increases.

Condition 6: It is assumed that:

$$\sup_i c_i < \infty,\ C_R < \infty,\ c_R(i) < \infty,\ w \leq i \leq m+1.$$

Let

$$D_k(i) = \sum_{j=k}^{m+1} p_{ij}(a), L_k(i) = \sum_{j=0}^{k} p_{ij}(a)$$

and

$$T_k(i) = \sum_{j=0}^{k} p_{ij}(a)$$

be functions related to transitions probabilities $p_{ij}(a)$.

Condition 7:

- For action a=0 and for a fixed k=0,1,...,m, it is assumed that:

$$D_k(i) \leq D_k(i+1),\, 0 \leq i \leq m.$$

- For action a=1A and for a fixed k=0,1,...,m-1, it is assumed that:

$$L_k(i) \geq L_k(i+1)\ \ ,\, 1 \leq i \leq m.$$

- For action a=1B and for a fixed k=0,1,...,m-1, it is assumed that:

$$T_k(i) \geq T_k(i+1),\, 1 \leq i \leq m.$$

The function $D_k(i) = \sum_{j=k}^{m+1} p_{ij}\,(a)$ indicates that if the system has a degree of deterioration i, and $\tilde{\imath} < i$, it is more likely to move to degrees of deterioration close to a degree i compared to $\tilde{\imath}$, if the system continues to operate. Furthermore, the monotonicity of $p_{ij}(0)$ indicates that it is more probably to move to state j rather than to state j+1. According to the definition of $L_k(i) = \sum_{j=0}^{k} p_{ij}\,(a)$ if the system has a degree of deterioration i, and $\tilde{\imath} > i$, it is more likely to move to lower degrees of deterioration close to a degree i compared to $\tilde{\imath}$ if the action of preventive maintenance is chosen for the system. The monotonicity $p_{ij}(1A)$ indicate that it is more likely to move to state j rather than to state j-1. According to the definition of $T_k(i) = \sum_{j=0}^{k} p_{ij}\,(a)$ if the system has a degree of deterioration i and $\tilde{\imath} > i$, it is more likely to move to lower degrees of deterioration close to a degree i compared to $\tilde{\imath}$, if the condition-based maintenance action is chosen for the system. The monotonicity of $p_{ij}(1B)$ indicate that it is more likely to move to state j-1 rather than to state j.

Condition 7 is an Increasing Failure Rate (IFR) assumption which it may be shown to be equivalent to the following one: For actions a∈{0,1A,1B} and for any function $h(j)$ which is non-decreasing (or non-increasing) in j, the function $\sum_{j=0}^{m+1} p_{ij}(a)h(j)$ is also non-decreasing (or non-increasing) in i.

Figure 1 and Table 3 summarize the main features of the model.

4. Mathematical models for determining the optimal policy

Let $\bar{C}_\alpha(i,a)$ be the expected discounted cost until the transition to the next state if the present state of the process is state i∈S, the discounted factor is equal to α>0 and action a∈{0,1A,1B,2} is chosen. For each action a∈{0,1A,1B,2} and for each state i∈S:

$$\bar{C}_\alpha(0,0) = c_0, \tag{1}$$

$$\bar{C}_\alpha(i,0) = c_i, \qquad 1 \leq i \leq m, \tag{2}$$

$$\bar{C}_\alpha(i,1A) = C_{PM} + \sum_{j=0}^{i-1} p_{ij}(1A) \int_0^\infty \int_0^t e^{-\alpha s} c_{PM}(i) ds dF_{ij}^{PM}(t)\,, \quad 1 \leq i \leq m, \tag{3}$$

$$\bar{C}_\alpha(i,1B) = C_{CBM} + \sum_{j=0}^{i-1} p_{ij}(1B) \int_0^\infty \int_0^t e^{-\alpha s} c_{CBM}(i) ds dF_{ij}^{CBM}(t)\,, \quad 1 \leq i \leq m, \tag{4}$$

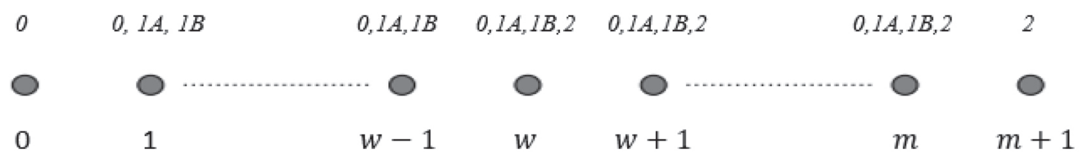

Figure 1. State-action graph of the proposed model

Table 3. A brief summary of the proposed model

Action in State i:	*Transition probability:*	*Costs:*
0: Leave the equipment to operate	$p_{ij}(0),\ j \geq i$	c_i
1A: Start preventive maintenance	$p_{ij}(1A),\ j < i$	C_{PM}, $c_{PM}(i)$
1B: Start condition-based maintenance	$p_{ij}(1B),\ j < i$	C_{CBM}, $c_{CBM}(i)$
2: Replace the equipment	$p_{i0}(2)$	C_R, $c_R(i)$

$$\bar{C}_\alpha(i,2) = C_R + \int_0^\infty \int_0^t e^{-\alpha s} c_R(i) ds dF_{i,0}^R(t),$$

$$w \le i \le m+1, \tag{5}$$

The minimum total expected discounted costs $V_\alpha(i,n)$, $i \in S$, $n=1,2,...$ with n periods left to the time horizon, when the current state is i, satisfy the following equations for each $n=1,2,...$:

$$V_\alpha(0,n) = \bar{C}_\alpha(0,0) + \alpha \sum_{j=0}^{m+1} p_{0j}(0) V_\alpha(j,n-1), \tag{6}$$

$$V_\alpha(i,n) = min\{V_\alpha(i,n,0), V_\alpha(i,n,1A), V_\alpha(i,n,1B)\},$$
$$1 \le i \le w-1, \tag{7}$$

$$V_\alpha(i,n) = min\{V_\alpha(i,n,0), V_\alpha(i,n,1A), V_\alpha(i,n,1B), V_\alpha(i,n,2)\},$$
$$w \le i \le m, \tag{8}$$

$$V_\alpha(m+1,n) = \bar{C}_\alpha(m+1,2) + \int_0^\infty e^{-\alpha t} V_\alpha(0,n-1) dF_{m+1,0}^R(t), \tag{9}$$

where,

$$V_\alpha(i,n,0) = \bar{C}_\alpha(i,0) + \alpha \sum_{j=0}^{m+1} p_{ij}(0) V_\alpha(j,n-1), \tag{10}$$

$$V_\alpha(i,n,1A) = \bar{C}_\alpha(i,1A) + \sum_{j=0}^{i-1} p_{ij}(1A) \int_0^\infty e^{-\alpha t} V_\alpha(j,n-1) dF_{ij}^{PM}(t), \tag{11}$$

$$V_\alpha(i,n,1B) = \bar{C}_\alpha(i,1B) + \sum_{j=0}^{i-1} p_{ij}(1B) \int_0^\infty e^{-\alpha t} V_\alpha(j,n-1) dF_{ij}^{CBM}(t), \tag{12}$$

and

$$V_\alpha(i,n,2) = \bar{C}_\alpha(i,2) + \int_0^\infty e^{-\alpha t} V_\alpha(0,n-1) dF_{i,0}^R(t), \tag{14}$$

A computational solution of the problem is possible by applying well-known algorithms in SMDP models. The description of the value-iteration algorithm and the description of the policy-iteration algorithm in steps, are presented below. Note that the value-iteration algorithm, in most cases, may be programmed easily. These algorithms find an optimal stationary policy and an approximation to the value of the minimum total expected discounted cost. A possible practical example of the proposed model could be the case of a car service. The interim service can be considered as a preventive maintenance action, whereas the full service can be considered as a condition-based maintenance action. The car can be completely replaced not necessarily because it is non-operative at all, but also when the degree of its deterioration exceeds a specific value.

Value-iteration algorithm:

Step 0. Choose $V_\alpha(i,0)$, $i \in S$, specify $\varepsilon > 0$ and set $n:=1$.

Step 1. For each state $i \in S$, compute the value function $V_\alpha(i,n)$ from the following recursive relation:

$$V_\alpha(i,n) = \min_{a \in A(i)} \left\{ \bar{C}_\alpha(i,a) + \sum_{j=0}^{\infty} p_{ij}(a) \int_0^\infty e^{-\alpha t} V_\alpha(j,n-1) dF_{ij}(t|a) \right\}$$

and determine $f^{(n)}$ as the stationary policy whose actions minimize the right-hand side of the above equation for all states $i \in S$.

Step 2. The algorithm is stopped with policy $f^{(n)}$, when

$$\min_{i \in S} |V_\alpha(i,n) - V_\alpha(i,n-1)| < \frac{\varepsilon(1-\alpha)}{2\alpha},$$

where $\alpha \in (0,1)$ is the discounted factor and $\varepsilon > 0$ is a pre-specified tolerance number. For example, $\varepsilon = 10^{-4}$. The minimum total expected discounted cost is approximated by the quantity $g = \min_{i \in S} V_\alpha(i,n)$ (or $g = \min_{i \in S} V_\alpha(i,n-1)$). Otherwise, move to Step 3.

Step 3. Set $n:=n+1$ and go to Step 1.

Policy-iteration algorithm:

Step 0. Choose an arbitrary initial trial policy R_1. Set $n:=1$.

Iteration n:

Step 1. For each state $i \in S$, and for action $a \in R_n$, solve the system of S equations:

$$V_\alpha(i,n) = \bar{C}_\alpha(i,a) + \sum_{j=0}^{\infty} p_{ij}(a) \int_0^\infty e^{-\alpha t} V_\alpha(j,n) dF_{ij}(t|a)$$

with unknown values $V_\alpha(i,n)$, $i \in S$.

Step 2. Using the values $V_\alpha(i,n)$, find a new policy R_{n+1}, by choosing $R_{n+1}(i) = a_i$, for each state $i \in S$, where a_i, is the decision that minimizes the quantity

$$\left\{ \bar{C}_\alpha(i,a) + \sum_{j=0}^{\infty} p_{ij}(a) \int_0^\infty e^{-\alpha t} V_\alpha(j,n) dF_{ij}(t|a) \right\}$$

So, for each state $i \in S$ determine the action a_i that:

$$\min_{a \in A(i)} \left\{ \bar{C}_\alpha(i,a) + \sum_{j=0}^{\infty} p_{ij}(a) \int_0^\infty e^{-\alpha t} V_\alpha(j,n) dF_{ij}(t|a) \right\}$$

Step 3. If for the new policy R_{n+1}, holds $R_{n+1} = R_n$, then the algorithm is stopped with policy R_{n+1}. Otherwise, set $n:=n+1$ and go to Step 1.

Although it seems difficult to analytically prove the form of the optimal policy, extensive numerical results indicate that the optimal policy is of a threshold type. The following conjecture concerning the form of the model optimal policy can be given.

Conjecture for the form of the optimal policy: There are two critical values i^* and i^{**}, such that the optimal policy prescribes the action of doing nothing for working equipment conditions i smaller than i^*, i.e. for conditions i such that $i<i^*$, prescribes the action of a type of equipment maintenance (preventive or condition-based maintenance), if the working equipment condition i is greater than or equal to i^* and smaller than i^{**} i.e. for conditions i such that $i^*\leq i<i^{**}$, and prescribes the action of equipment replacement if its working condition i is greater than or equal to i^{**}, i.e. for conditions i such that $i\geq i^{**}$.

5. A linear programming formulation

A linear programming formulation is used in this section to solve the minimum total expected discount cost problem. The basic advantage of such a method is the possibility of introducing constraints into certain system parameters which is very common in real-life models. For example, in a maintenance problem, constraints may be placed on the fraction of time the system is in repair or a type of maintenance. According to the approach of Puterman [25], the following formulation can be utilized to solve the total expected discounted cost problem for a semi-Markov decision process:

$$\text{Minimize } z = \sum_{i\epsilon S}\sum_{a\epsilon A(i)} \bar{C}_\alpha(i,a)\cdot y_{ia} \tag{15}$$

subject to:

$$\sum_{a\epsilon A(j)} y_{ja} - \left(\sum_{i\epsilon S}\sum_{a\epsilon A(i)} m_{ij}(a)\cdot y_{ia}\right) = b_j,\ j\epsilon S,$$
$$y_{ia}\geq 0,\quad i\epsilon S, a\epsilon A(i), \tag{16}$$

where,

$$m_{ij}(a) = p_{ij}(a)\int_0^\infty e^{-\alpha t}\, dF_{ij}(t|a).$$

The parameters $b_j>0$ must satisfy the condition $\sum_{i\epsilon S} b_i=1$, and they can be chosen arbitrarily. Their choice affects the optimal value of z, not the resulting optimal policy. If they are chosen such that $b_j=P\{X_0=j\}$, then the variable y_{ia} can be interpreted as the expected discounted time of being in state i and making decision a, and z can be interpreted as the corresponding expected total discounted cost. So, linear programming formulation for the model can be given as follows:

$$\text{Minimize } z = \sum_{i\epsilon S}\sum_{a\epsilon A(i)} \bar{C}_\alpha(i,a)\cdot y_{ia}$$

subject to:

$$\sum_{a\epsilon A(j)} y_{ja} - \big(a\cdot p_{0j}(0)y_{00} +$$
$$\sum_{i=1}^{m}(a\cdot p_{ij}(0)\,y_{i0} + m_{ij}(1A)y_{i,1A} + m_{ij}(1B)y_{i,1B}) +$$
$$\sum_{i=w}^{m} m_{ij}(2)y_{i2} + m_{m+1,j}(2)y_{m+1,2}\big) = b_j, j\in S,$$

$$y_{ia}\geq 0,\ i\epsilon S, a\epsilon A(i).$$

In Example 3 of the following section, the linear programming formulation of the model is numerically implemented.

6. Numerical results

Three numerical examples are presented in this part. In Example 1, it is assumed that the preventive maintenance, the condition-based maintenance, and the replacement equipment times are exponentially distributed, and in Example 2, it is assumed that they follow the Gamma distribution, respectively. In Example 3, a linear programming implementation for the model is also presented.

Example 1 Suppose that:

$$m=10,\ C_{PM}=4,\ C_{CBM}=4.2,\ C_R=5,\ c_R(i)=1.2,$$
$$w\leq i\leq m+1,\ c_i=1.5(i+1),\ 0\leq i\leq m,$$
$$c_{PM}(i)=0.09(i+1),\ 1\leq i\leq m,$$
$$c_{CBM}(i)=0.1(i+1),\ 1\leq i\leq m,\ \alpha=0.4,\ \varepsilon=0.01,\ w=7.$$

The transition probabilities are assumed to be given by:

$$p_{ij}(0) = \frac{m+2-j}{\sum_{q=0}^{m+2-i} q},\ 0\leq i\leq m,\qquad i\leq j\leq m+1,$$

$$p_{ij}(1A) = \frac{j+1}{\sum_{q=0}^{i} q},\ 1\leq i\leq m,\qquad 0\leq j\leq i-1$$

and

$$p_{ij}(1B) = \frac{i-j}{\sum_{q=0}^{i} q},\ 1\leq i\leq m,\qquad 0\leq j\leq i-1.$$

It is also supposed that preventive maintenance, condition-based maintenance and equipment replacement times are exponentially distributed with means $m_{PM}=0.833$, $m_{CBM}=1$ and $m_R=1.666$, respectively. In the following matrices, transition probabilities $p_{ij}(0)$, $p_{ij}(1A)$, $p_{ij}(1B)$ and the numerical implementation of the functions D_k, L_k and T_k are presented.

Table 4. Transition probability matrix $p_{ij}(0)$

$p_{ij}(0)$	0	1	2	3	4	5	6	7	8	9	10	11
0	0.153	0.141	0.128	0.115	0.102	0.089	0.076	0.064	0.051	0.038	0.025	0.012
1	0	0.166	0.151	0.136	0.121	0.106	0.090	0.075	0.060	0.045	0.030	0.015
2	0	0	0.181	0.163	0.145	0.127	0.109	0.090	0.072	0.054	0.036	0.018
3	0	0	0	0.2	0.177	0.155	0.133	0.111	0.088	0.066	0.044	0.022
4	0	0	0	0	0.222	0.194	0.166	0.138	0.111	0.083	0.055	0.027
5	0	0	0	0	0	0.250	0.214	0.178	0.142	0.107	0.071	0.035
6	0	0	0	0	0	0	0.285	0.283	0.190	0.142	0.095	0.047
7	0	0	0	0	0	0	0	0.333	0.266	0.200	0.133	0.066
8	0	0.	0	0	0	0	0	0	0.400	0.300	0.200	0.100
9	0	0	0	0	0	0	0	0	0	0.500	0.333	0.166
10	0	0	0	0	0	0	0	0	0	0	0.666	0.333
11	0	0	0	0	0	0	0	0	0	0	0	0

Table 5. Transition probability matrix $p_{ij}(1A)$

$p_{ij}(1A)$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1.000	0	0	0	0	0	0	0	0	0	0	0
2	0.333	0.666	0	0	0	0	0	0	0	0	0	0
3	0.166	0.333	0.500	0	0	0	0	0	0	0	0	0
4	0.100	0.200	0.300	0.400	0	0	0	0	0	0	0	0
5	0.066	0.133	0.200	0.266	0.033	0	0	0	0	0	0	0
6	0.047	0.095	0.142	0.190	0.238	0.285	0	0	0	0	0	0
7	0.035	0.071	0.107	0.142	0.178	0.214	0.250	0	0	0	0	0
8	0.027	0.055	0.083	0.111	0.138	0.166	0.194	0.222	0	0	0	0
9	0.022	0.044	0.066	0.088	0.111	0.133	0.155	0.177	0.200	0	0	0
10	0.018	0.036	0.054	0.072	0.090	0.109	0.127	0.145	0.163	0.181	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0

Table 6. Transition probability matrix $p_{ij}(1B)$

$p_{ij}(1B)$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1.000	0	0	0	0	0	0	0	0	0	0	0
2	0.666	0.333	0	0	0	0	0	0	0	0	0	0
3	0.500	0.333	0.166	0	0	0	0	0	0	0	0	0
4	0.400	0.300	0.200	0.100	0	0	0	0	0	0	0	0
5	0.333	0.266	0.200	0.133	0.066	0	0	0	0	0	0	0
6	0.285	0.238	0.190	0.142	0.095	0.047	0	0	0	0	0	0
7	0.250	0.214	0.178	0.142	0.107	0.071	0.035	0	0	0	0	0
8	0.222	0.194	0.166	0.138	0.111	0.083	0.055	0.027	0	0	0	0
9	0.200	0.177	0.155	0.133	0.111	0.088	0.066	0.044	0.022	0	0	0
10	0.181	0.163	0.145	0.127	0.109	0.090	0.072	0.054	0.036	0.018	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0

On Table 5, note that if the system is in i=3 and chooses preventive maintenance, it is more likely to go to state 2, then to state 1 and with less probability to state 0. On the contrary, on Table 6, note that if the system is in i=3 and chooses the condition-based maintenance action, then it is more likely to go to state 0, then to state 1, and less likely to state 2.

Table 7. The functions D_k and L_k

$D_k(i)$	i=0	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
k=0	1	1	1	1	1	1	1	1	1	1	1
k=1	0.846	1	1	1	1	1	1	1	1	1	1
k=2	0.705	0.833	1	1	1	1	1	1	1	1	1
k=3	0.577	0.682	0.818	1	1	1	1	1	1	1	1
k=4	0.461	0.545	0.654	0.800	1	1	1	1	1	1	1
k=5	0.359	0.424	0.509	0.622	0.777	1	1	1	1	1	1
k=6	0.269	0.318	0.381	0.466	0.583	0.750	1	1	1	1	1
k=7	0.192	0.227	0.272	0.333	0.416	0.535	0.714	1	1	1	1
k=8	0.128	0.151	0.181	0.222	0.277	0.357	0.476	0.666	1	1	1
k=9	0.076	0.091	0.109	0.133	0.166	0.214	0.285	0.400	0.600	1	1
k=10	0.038	0.045	0.054	0.066	0.083	0.107	0.142	0.200	0.300	0.500	1
k=11	0.012	0.015	0.018	0.022	0.027	0.035	0.047	0.066	0.100	0.166	0.333

$L_k(i)$	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
k=0	1	0.333	0.166	0.100	0.066	0.047	0.035	0.027	0.022	0.018
k=1	1	1	0.5	0.3	0.2	0.142	0.107	0.083	0.066	0.054
k=2	1	1	1	0.6	0.4	0.285	0.214	0.166	0.133	0.109
k=3	1	1	1	1	0.666	0.476	0.357	0.277	0.222	0.181
k=4	1	1	1	1	1	0.714	0.535	0.416	0.333	0.272
k=5	1	1	1	1	1	1	0.75	0.583	0.466	0.381
k=6	1	1	1	1	1	1	1	0.777	0.622	0.509
k=7	1	1	1	1	1	1	1	1	0.8	0.654
k=8	1	1	1	1	1	1	1	1	1	0.818
k=9	1	1	1	1	1	1	1	1	1	1

In Table 7, according to the values of the function D, if the system is in state $\tilde{\imath}=3$ the cumulative probability (0.3333) of going to state 7 up to state 11 is smaller than being in state i=4 and going to the same states (0.4165). According to the function L, if the system is in state i=7, the cumulative probability of going to state 0 up to state 2 (0.2142) is greater than being in $\tilde{\imath}=8$ and going to the same states (0.1667). Similar conclusions related to the function L are also observed for function T in Table 8.

In the next table, the form of optimal policy is presented. The value-iteration algorithm gave the ε-optimal policy after 15 iterations. In states 0,1 and 2, the system continues its operation without any intervention. The condition-based maintenance of the equipment is selected for the system in states 3 up to 6, and in states 7 up to 10 the system returns to state 0 by replacing the equipment. The structure of $p_{ij}(a)$ and the values of c_i, $c_{PM}(i)$, $c_{CBM}(i)$, $c_R(i)$, C_{PM}, C_{CBM} ensure this behavior of the system.

Table 8. The function T_k

$T_k(i)$	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
k=0	1	0.666	0.500	0.400	0.333	0.285	0.250	0.222	0.200	0.181
k=1	1	1	0.833	0.700	0.600	0.523	0.464	0.416	0.377	0.345
k=2	1	1	1	0.900	0.800	0.714	0.642	0.583	0.533	0.490
k=3	1	1	1	1	0.933	0.857	0.785	0.722	0.666	0.618
k=4	1	1	1	1	1	0.952	0.892	0.833	0.777	0.727
k=5	1	1	1	1	1	1	0.964	0.916	0.866	0.818
k=6	1	1	1	1	1	1	1	0.972	0.933	0.890
k=7	1	1	1	1	1	1	1	1	0.977	0.945
k=8	1	1	1	1	1	1	1	1	1	0.981
k=9	1	1	1	1	1	1	1	1	1	1

Table 9. Optimal policy form

State	0	1	2	3	4	5	6	7	8	9	10	11
Action	0	0	0	1B	1B	1B	1B	2	2	2	2	2

Table 10. Optimal policy as C_{PM} varies and the corresponding values of g

State	0	1	2	3	4	5	6	7	8	9	10	11	
C_{PM}=2.0	0	1A	1A	1A	1A	1A	1A	1A	1A	1A	1A	2	g=3.88
C_{PM}=2.5	0	1A	1A	1A	1A	1A	1A	1A	2	2	2	2	g=4.28
C_{PM}=3.0	0	0	1A	1A	1A	1A	1B	2	2	2	2	2	g=4.55
C_{PM}=3.5	0	0	1A	1B	1B	1B	1B	2	2	2	2	2	g=4.62
C_{PM}=4.0	0	0	0	1B	1B	1B	1B	2	2	2	2	2	g=4.64

Table 11. Optimal policy as c_i varies and the corresponding values of g

State	0	1	2	3	4	5	6	7	8	9	10	11	
c_i=0.5(i+1)	0	0	0	0	0	0	0	0	0	2	2	2	g=2.38
c_i=1.0(i+1)	0	0	0	0	1B	1B	1B	2	2	2	2	2	g=3.70
c_i=1.5(i+1)	0	0	0	1B	1B	1B	1B	2	2	2	2	2	g=4.64
c_i=2.0(i+1)	0	0	1B	1B	1B	1B	1B	2	2	2	2	2	g=5.45
c_i=2.5(i+1)	0	1A	1B	1B	1B	1B	1B	2	2	2	2	2	g=6.21

Table 12. Optimal policy as C_R varies and the corresponding values of g

State	0	1	2	3	4	5	6	7	8	9	10	11	
C_R=5.0	0	0	0	1B	1B	1B	1B	2	2	2	2	2	g=4.64
C_R=5.5	0	0	0	1B	1B	1B	1B	2	2	2	2	2	g=4.70
C_R=6.0	0	0	0	1B	1B	1B	1B	1B	2	2	2	2	g=4.76
C_R=6.5	0	0	0	1B	1B	1B	1B	1B	1B	1B	2	2	g=4.78
C_R=7.0	0	0	0	1B	1B	1B	1B	1B	1B	1B	1B	2	g=4.80

According to the conjecture concerning the form of the optimal policy, from the above table, the two critical values i^* and i^{**} of the optimal policy are equal to $i^*=3$ and $i^{**}=7$, respectively. Many numerical examples provide strong indication that optimal policy is of a threshold type. For the data of Example 1, and $m=100$, $w=75$ the optimal policy is characterized by the critical values $i^*=2$ and $i^{**}=76$. In the following tables, the optimal policy is modified for a variety of variables. In Table 10, the optimal policy for different values of C_{PM} and minimum total expected discounted cost for each policy is presented. For low values of C_{PM}, the action of preventive maintenance is selected for the system in most states. As the value of C_{PM} increases, preventive maintenance becomes unprofitable, and the system selects the action of condition-based maintenance and the action of equipment replacement where it is possible, due to low replacement costs.

In Table 11, the optimal policy for different values of c_i and the corresponding values g for each policy is presented.

As it can be seen from Table 11, as c_i increases, the system chooses less and less the action of leaving the equipment to operate and chooses condition-based maintenance and the action of equipment replacement. There is interested in the value $c_i=2.5(i+1)$, $1 \leq i \leq m$, where, in state 1 the system selects preventive maintenance. This is reasonable because both maintenance actions lead the system to state 0, so the one with the lowest cost is selected. In Table 12, the form of optimal policy for different values of C_R and the corresponding values g of the minimum total expected discounted cost for each policy is presented.

According to Table 12 above, the increment of C_R leads the system to the choice of condition-based maintenance action instead of the action of equipment replacement.

Example 2 Suppose that:

$$m=10,\ C_{PM}=4,\ C_{CBM}=4.2,\ C_R=5,\ c_R(i)=1.2,$$
$$w\le i\le m+1,\ c_i=1.5(i+1),\ 0\le i\le m,$$
$$c_{PM}(i)=0.09(i+1),\ 1\le i\le m,\ c_{CBM}(i)=0.1(i+1),$$
$$1\le i\le m,\ \alpha=0.4,\ \varepsilon=0.01,\ w=7.$$

The transition probabilities are given as in Example 1. It is also assumed that the preventive maintenance, the condition-based maintenance, and the equipment replacement times follow the Gamma distribution with shape parameters equal to $a_1, a_1', a_2 > 0$ and scale parameters equal to $b_1, b_1', b_2 > 0$, respectively. It is assumed that the shape parameters are: $a_1=2$, $a_1'=2$ and $a_2=3$ and the scale parameters are: $b_1=0.42$, $b_1'=0.5$ and $b_2=0.55$, respectively. Then, $m_{PM}=0.84$, $m_{CBM}=1$ and $m_R=1.65$. In the next table, the optimal policy is presented. Note that is sustained the same as in Example 1.

In Tables 14, 15 and 16 the form of optimal policy for different values of C_{PM}, c_i and C_R respectively and the corresponding values of g for each policy is presented. The evolution of optimal policy and the changes into the values of g are similar as in Example 1.

In the following example, the linear programming formulation of the model is numerically implemented. Appropriate constraints are introduced, and the minimum cost is estimated under these constraints.

Example 3 Suppose that:

$$m=10,\ C_{PM}=4,\ C_{CBM}=4.2,\ C_R=5,\ c_R(i)=1.2,$$
$$w\le i\le m+1,\ c_i=1.5(i+1),\ 0\le i\le m,$$
$$c_{PM}(i)=0.09(i+1),\ 1\le i\le m,\ c_{CBM}(i)=0.1(i+1),$$
$$1\le i\le m,\ \alpha=0.4,\ \varepsilon=0.01,\ w=7.$$

The transition probabilities are given as in Example 1. It is also assumed that the preventive maintenance, the condition-based maintenance, and the equipment replacement times are exponentially distributed with mean equal to $m_{PM}=0.833$, $m_{CBM}=1$ and $m_R=1.666$, respectively. The parameters b_j, $j\in S$ are arbitrarily chosen to be equal to $\frac{1}{12}$. For each state $i\in S$, the action $R_i=a$, is defined for the actions a such

Table 13. Optimal policy form

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Action	*0*	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*

Table 14. Optimal policy as C_{PM} varies and the corresponding values of g

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	
$C_{PM}=2.0$	*0*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*2*	*g=3.82*
$C_{PM}=2.5$	*0*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*1A*	*2*	*2*	*2*	*g=4.22*
$C_{PM}=3.0$	*0*	*0*	*1A*	*1A*	*1A*	*1A*	*1A*	*2*	*2*	*2*	*2*	*2*	*g=4.50*
$C_{PM}=3.5$	*0*	*0*	*1A*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=4.60*
$C_{PM}=4.0$	*0*	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=4.61*

Table 15. Optimal policy as c_i varies and the corresponding values of g

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	
$c_i=0.5(i+1)$	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*2*	*2*	*2*	*g=2.38*
$c_i=1.0(i+1)$	*0*	*0*	*0*	*0*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=3.68*
$c_i=1.5(i+1)$	*0*	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=4.61*
$c_i=2.0(i+1)$	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=5.39*
$c_i=2.5(i+1)$	*0*	*1A*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=6.07*

Table 16. Optimal policy as C_R varies and the corresponding values of g

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	
$C_R=5.0$	*0*	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=4.61*
$C_R=5.5$	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*	*g=4.67*
$C_R=6.0$	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*g=4.72*
$C_R=6.5$	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*g=4.74*
$C_R=7.0$	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*1B*	*2*	*g=4.75*

that y_{ia}>0. The values of y_{ia} after the implementation of the linear programming algorithm, are given below:

$$y_{00}=0.5838, y_{10}=0.2510, y_{20}=0.2211,$$
$$y_{3,1B}=0.1736, y_{4,1B}=0.1467, y_{5,1B}=0.1303,$$
$$y_{6,1B}=0.1201, y_{72}=0.1139, y_{82}=0.1078,$$
$$y_{92}=0.1017, y_{10,2}=0.0956, y_{11,2}=0.0895$$

and the others y_{ia} are equal to zero. The above values of y_{ia} yield the optimal policy presented in the following table.

The value of objective function for optimal solution equals to z=8.3864. This value is closely related to the values b_j and the values of V_α (i,n) for this optimal policy. It was obtained by the value iteration algorithm, since $z = \sum_{i\epsilon S} V_\alpha(i,n) \cdot b_i$. A policy d is called a stationary randomized policy when it is described by a probability distribution: $\left\{d_\alpha(i) = \frac{y_{ia}}{\sum_{b\epsilon A(i)} y_{ib}} : a\epsilon A(i)\right\}$ for each state $i\epsilon S$. Under policy d action $a\epsilon A(i)$ is chosen with probability $d_\alpha(i)$ whenever the process is in state i. If $d_\alpha(i)$ is 0 or 1 for every i and $a\epsilon A(i)$, the stationary randomized policy d reduces to the deterministic policy. Because of the addition of constraints, the optimal policy may be randomized.

It is supposed that in states 1,2,3,4,5,6 the expected discounted time of condition-based maintenance is no more than 0.3, which corresponds to the constraint:

$$y_{1,1B}+y_{2,1B}+y_{3,1B}+y_{4,1B}+y_{5,1B}+y_{6,1B}\leq 0.3,$$

and in states 7,8,9,10,11 the expected discounted time of equipment replacement is no more than 0.2, which corresponds to the constraint:

$$y_{72}+y_{82}+y_{92}+y_{10,2}+y_{11,2}\leq 0.2.$$

The above constraints are added to the previous linear program and the new linear program has the following optimal solution:

$$y_{00}=0.3906, y_{10}=0.3137, y_{20}=0.3603,$$
$$y_{2,1A}=0.2520, y_{4,1A}=0.1632, y_{4,1B}=0.0138,$$
$$y_{5,1B}=0.1520, y_{6,1B}=0.1342, y_{7,1B}=0.1210,$$
$$y_{8,1B}=0.1108, y_{9,1B}=0.0891, y_{92}=0.0138,$$
$$y_{10,2}=0.0964,\ y_{11,2}=0.0898$$

and the others y_{ia} are equal to zero. The value of the objective function for the optimal solution is now equal to z=9.1014. Because of the additional constraints imposed, it is reasonable to have a higher value of the objective function for the optimal solution, than the one obtained without them. The above linear program solution that corresponds to the randomized policy is given in the following table.

In state 4, preventive maintenance is done with probability 0.92 and condition-based maintenance is done with probability 0.08. In state 9, a condition-based maintenance is selected with probability 0.86 and the action of equipment replacement is selected with probability 0.14.

7. Comparison of two models

In this section, we compare the model described in Section 2 with a simplified model that has the possibility only for preventive maintenance and the equipment is replaced only in the degree of deterioration m+1, which is the most common practice in scheduling system equipment maintenance.

In diagrams into Figure 2 below, the comparison of our basic model described in Section 2, with the simplified model, for values $C_{CBM}\in\{4.2,4.9\}$ when $C_{PM}\in[2,4]$ in the cases in which w=7, w=9 and w=11 is presented. In these diagrams, the preventive maintenance, the condition-based maintenance, and the equipment replacement times are exponentially distributed.

For small values of C_{PM}, the two models work similarly. The system chooses the preventive maintenance where it is possible, so the cost for both models is equal. As the values of C_{PM} increase, the condition-based maintenance option becomes preferable for the basic model and the system selects the equipment replacement in some states, so the cost

Table 17. The optimal policy after the implementation of the linear programming algorithm

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Action	*0*	*0*	*0*	*1B*	*1B*	*1B*	*1B*	*2*	*2*	*2*	*2*	*2*

Table 18. Optimal policy after the implementation of the linear programming algorithm with additional constraints into condition-based maintenance and replacement times

State	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Action	*0*	*0*	*0*	*1A*	*1A* *1B*	*1B*	*1B*	*1B*	*1B*	*1B* *2*	*2*	*2*

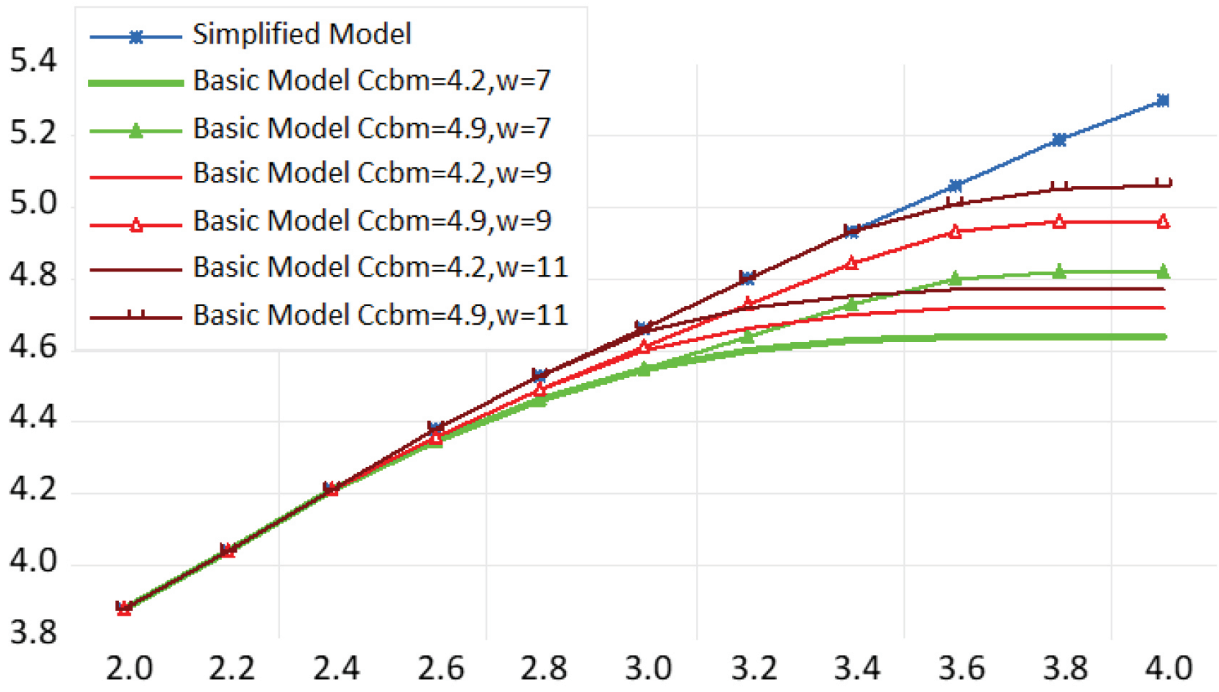

Figure 2. Comparison between the two models with exponentially distributed times

in basic model is lower. As the value of w increases, the equipment replacement is chosen in fewer states, and this is the reason that causes an increment in the cost of the basic model. In the case in which w=11, where the choice of replacement does not exist, the basic model has almost the same cost as the simplified one. Based on this analysis, the choice of basic model for the management of the system is more efficient. This model includes all the options of the simplified model, so it can take advantage of the cost values and make the choice for the type of maintenance each time. The choice in the value of w does not affect the choice in the suitability of the model and the cost in the basic model is always less than or equal to the cost of the simplified model.

8. Conclusions and further research

In this work, a semi-Markov decision process model for the maintenance of system equipment which contains two types of maintenance, and the possibility that the system can define the replacement of the equipment if the degree of its deterioration exceeds a specific value, is presented. There is strong numerical verification that the resulting optimal policies are of a threshold type, and they are characterized by two critical values. Using an appropriate linear programming formulation, constraints are introduced on system parameters and in these cases, the changes into the form of the optimal policy are observed. The linear programming formulation to solve the problem of expected discounted cost for a semi-Markov decision process is something that has not previously been seen in the literature. Finally, the possibility to choose between different types of maintenance and replacement of equipment in degrees of deterioration other than the degree of deterioration equal to m+1, is an advantage for the system. The reason is that the system can benefit from low prices of some type of maintenance or replacement and make the most ideal action choices for each state.

The described model in this paper is limited to single-equipment systems. However, there are multiple-equipment systems (for example, parallel machine processing systems) in which their efficiency depends on the maintenance management of each equipment. The generalization of the model and its adaptation to systems with more than one piece of equipment is a possible direction for future research. The construction of a two-dimensional semi-Markov decision model in which two pieces of equipment of a system could be maintained or completely replaced is an extension and an area of further research. For such a model, the working condition of the system can be described by the pair (a_1, a_2), where a_1=0,1,..,m_i+1, i=1,2, is the m_i+2 degrees of deterioration for each equipment $i\in\{1,2\}$. The possible actions for this model could be (a) to maintain one equipment and the other to continue its operation,

(b) to replace one equipment and the other to continue its operation, (c) to replace one equipment and the other to be maintained, (d) to replace both equipment, (e) to maintain both equipment, (f) to continue the operation of both equipment.

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